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A REMARK ON α-CONVEX FUNCTIONS

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ABSTRACT. Let α be real and suppose that $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is analytic in the unit disk Δ. If $\mathrm{Re}[(1-\alpha)zf'(z)/f(z)+\alpha(1+zf''(z)/f'(z))] > 0$ for $z \in \Delta$, then $f(z)$ is said to be α-convex function. In this paper, we will show that if an α-convex function $f(z)$ satisfies certain conditions, then $f(z)$ is starlike of order at least 1/2.

1. Introduction. Let A be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

which are analytic in $\Delta = \{z : |z| < 1\}$.

A function $f(z)$ in A is said to be starlike iff

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0 \qquad \text{in } \Delta .$$

Further, a function $f(z)$ in A is said to be convex iff

$$1 + \mathrm{Re}\,\frac{zf''(z)}{f'(z)} > 0 \qquad \text{in } \Delta .$$

It is well known that all convex functions are starlike of order at least 1/2 [2,5].

On the other hand, a function $f(z)$ in A is said to be α-convex iff

$$\mathrm{Re}\left[\,(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right)\right] > 0 \qquad \text{in } \Delta .$$

Miller, Mocanu and Reade [4] proved the following theorem.

THEOREM A. If $f(z) \in A$ is α-convex in Δ, then $f(z)$ is starlike in Δ. Moreover, if $\alpha \geqq 1$, then $f(z)$ is convex in Δ, and if $\alpha \leqq -1$, then $1/f(1/z)$ is convex for $|z| > 1$.

It is the purpose of the present paper to partly improve THEOREM A.

2. Main theorem. We need the following lemma.

LEMMA 1. Let $w(z)$ be analytic in Δ, $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z|=r<1$ at a point z_0, then we can write

$$z_0 w'(z_0) = k w(z_0)$$

where k is a real number and $k \geqq 1$.

We owe this lemma to Jack [1] (also, by Miller and Mocanu [3]).

LEMMA 2. Let $p(z)$ be analytic in Δ, $p(0) = 1$ and suppose that

(1) $$\mathrm{Re}\left(p(z) + \alpha \frac{zp'(z)}{p(z)} \right) > \frac{1-\alpha}{2} \qquad \text{in } \Delta,$$

when α is a positive real number, or

(2) $$\mathrm{Re}\left(p(z) + \alpha \frac{zp'(z)}{p(z)} \right) < \frac{1-\alpha}{2} \qquad \text{in } \Delta,$$

when $\alpha < -1$.

Then we have

$$\left| \frac{p(z) - 1}{p(z)} \right| < 1 \qquad \text{in } \Delta,$$

or

$$\mathrm{Re}\, p(z) > \frac{1}{2} \qquad \text{in } \Delta.$$

PROOF. From the assumptions (1) and (2), we have $p(z) \neq 0$ in Δ, because if there exists a point $\beta \in \Delta$ such that $p(\beta) = 0$ and $\beta \neq 0$, then we can write

$$p(z) = (z - \beta)^s p_1(z),$$

where s is a positive integer and $p_1(\beta) \neq 0$, then we have

(3) $$\mathrm{Re}\left[p(z) + \alpha \frac{zp'(z)}{p(z)} \right]$$

$$= \mathrm{Re}\left[(z-\beta)^s p_1(z) + \frac{\alpha s z}{z - \beta} + \frac{\alpha z p_1'(z)}{p_1(z)} \right].$$

Letting $z \to \beta$ on the straight line which pass through the origin and β, then the right hand side of (3) become positive and negative infinite.

This contradicts (1) and (2).

Therefore we have

$$p(z) \neq 0 \quad \text{in } 0<|z|<1.$$

On the other hand, from the assumption $p(0) = 1$, this shows that

$$p(z) \neq 0 \quad \text{in } \Delta .$$

Let us put

$$p(z) = \frac{1}{1 - w(z)}$$

or

$$w(z) = 1 - \frac{1}{p(z)} .$$

Then $w(z)$ is analytic in Δ and $w(0) = 0$, since $p(z) \neq 0$ in Δ. If there exists a point z_0 such that $|w(z)|<1$ for $|z|<|z_0|<1$, $|w(z_0)| = 1$, $w(z_0) = e^{i\theta}$, then from LEMMA 1, we have

$$z_0 w'(z_0) = kw(z_0).$$

Then we have

$$\begin{aligned}
\operatorname{Re}\left(p(z_0) + \frac{z_0 p'(z_0)}{p(z_0)} \right) &= \operatorname{Re}\left(\frac{1}{1 - w(z_0)} - \frac{\alpha z_0 w'(z_0)}{1 - w(z_0)} \right) \\
&= \operatorname{Re}\left(\frac{1}{1 - e^{i\theta}} + \frac{\alpha k e^{i\theta}}{1 - e^{i\theta}} \right) \\
&= \frac{1}{2} - \frac{\alpha k}{2} \begin{cases} \leqq \dfrac{1-\alpha}{2} & \text{for the case } \alpha > 0 \\[2mm] \geqq \dfrac{1-\alpha}{2} & \text{for the case } \alpha < -1. \end{cases}
\end{aligned}$$

This contradicts (1) and (2). Therefore we have $|w(z)|<1$ in Δ.

This shows that

$$\left| \frac{p(z) - 1}{p(z)} \right| < 1 \qquad \text{in } \Delta$$

or

$$\operatorname{Re} p(z) > \frac{1}{2} \qquad \text{in } \Delta .$$

From LEMMA 2, we easily have the following theorem.

MAIN THEOREM. Let $f(z) \in A$ and suppose that

$$\operatorname{Re}\left[(1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) \right]$$

$$\begin{cases} \geqq \dfrac{1-\alpha}{2} & \text{for the case } \alpha > 0 \\[2mm] \leqq \dfrac{1-\alpha}{2} & \text{for the case } \alpha < -1. \end{cases}$$

Then f(z) is starlike of order at least 1/2 and

$$\frac{|zf'(z) - f(z)|}{|zf'(z)|} < 1 \quad \text{in } \Delta .$$

REMARK. It is trivial that MAIN THEOREM partly improves THEOREM A.

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